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Journal of Geometry and Physics 25 (1998) 227–244

JOURNAL OF GEOMETRY AND PHYSICS

Noncommutative differential geometry, and the matrix representations of generalised algebras

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Received 19 November 1996; received in revised form 7 April 1997

Abstract

The underlying algebra for a noncommutative geometry is taken to be a matrix algebra, and the set of derivatives the adjoint of a subset of traceless matrices. This is sufficient to calculate the dual 1-forms, and show that the space of 1-forms is a free module over the algebra of matrices. The concept of a generalised algebra is defined and it is shown that this is required in order for the space of 2-forms to exist. The exterior derivative is generalised for higher-order forms and these are also shown to be free modules over the matrix algebra. Examples of mappings that preserve the differential structure are given. Also given are four examples of matrix generalised algebras, and the corresponding noncommutative geometries, including the cases where the generalised algebra corresponds to a representation of a Lie algebra or a q-deformed algebra. © 1998 Elsevier Science B.V.

Subj. Class.: Noncommutative geometry
1991 MSC: 17A30, 58A10
Keywords: Noncommutative geometry; Generalised algebra

1. Introduction

To define a noncommutative geometry or differential calculus, it is first necessary to introduce an algebra $\mathcal{A}$ that will replace the algebra of functions. There is a unique universal differential calculus for which all calculi are quotients. There are several methods of defining the quotient map necessary for the space of 1-forms. The method we use follows [3] by constructing it with respect to a subspace $\mathcal{B}$ of $\mathcal{A}$.

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PII S0393-0440(97)00027-2

In the early days [1,2] $\mathcal{B}$ was taken to be $\mathcal{A}$ itself. Later [9, Chap. 3] examples where $\mathcal{B}$ formed a Lie algebra, or some other algebraic relationship such as $[p, x] = 1$ as in quantum mechanics, or $xy = qyx$ as in q-deformed algebras, were studied.

For each subspace $\mathcal{B}$ one could construct a co-frame. This co-frame is loosely analogous to the orthonormal co-frame used in normal differential geometry. By quotienting the universal calculus one could then construct the set of 2-forms and higher-order forms.

It was discovered that if $\mathcal{B} = \mathcal{A}$, or $\mathcal{B}$ formed a Lie algebra or a quantum algebra then one could consistently impose the condition that the co-frame basis elements of the exterior algebra anticommute. Whilst for q-deformed algebras the basis elements of the exterior algebra q-anticommute.

It is only recently [3,4] that people have looked at a general $\mathcal{B}$. They showed that in order for forms of order 2 and above to exist puts constraints on the elements of $\mathcal{B}$. However, these constraints have not been pursued.

In this paper we impose the condition $\mathcal{A} = M_m(\mathbb{C})$ and that all the elements in $\mathcal{B}$ are traceless. In Section 3, we show that this is a sufficient condition for the co-frame to exist. However, this condition is not a necessary condition for the co-frame to exist. To show this we give some examples where $\mathcal{A} \neq M_n(\mathbb{C})$, some of which have a co-frame and others which do not. Since $M_m(\mathbb{C})$ is a finite approximation to the infinite-dimensional space of functions it is hoped that this procedure can be used as an alternative to the theory of renormalisation or lattice QFT.

In Section 2 we introduce the concept of a "generalised algebra". This is an algebraic structure that includes commutative algebras, anti-commutative algebras, Lie algebras, Clifford algebras and q-deformed algebras as examples. Each generalised algebra has a specific rank and the space of 2-forms is a free module over $\mathcal{A}$ of rank equal to the rank of the generalised algebra. In Section 4 we show that for 2-forms and higher forms to exist $\mathcal{B}$ must form a generalised algebra. In Section 5, we then give the structure of the higher-order forms, all of which are also free modules over $\mathcal{A}$, and an explicit expression for the exterior derivative. In Section 7 we give a couple of simple examples of maps between generalised algebras which are d-homomorphism, i.e. they preserve the differentiable structure.

To elucidate the relationship between the generalised algebra of $\mathcal{B}$ and the space of 2-forms we give, in Section 8, four examples: Much emphasis has been placed on the case that $\mathcal{B}$ form a Lie algebra. Especially since $su(2)$ corresponds to the fuzzy or noncommutative version of the sphere [9, Chap. 7.2] and $su(4)$ is an analogue of the Euclidianised compactified Minkowski space [8]. Another example is that of the q-deformed algebra, this has a finite-dimensional representation only if there exists an $m \in \mathbb{Z}$ such that $q^m = 1$. Finally a $\mathcal{B}$ is given of dimension 3 and rank 1 which may be thought of as the fuzzy ellipse.

For further references, and history of this subject the reader is asked to read the book [9].

1.1. Note on notation

Unless otherwise stated $\mathcal{A} = M_m(\mathbb{C})$. $\mathcal{B} \subset \mathcal{A}$ is a subspace of dimension n of traceless matrices and λ_a is a basis for $\mathcal{B}$. Early Roman letters used as indices $a, b, \ldots$ run over $1, \ldots, n$, and we use the Einstein summation convention so that the summation is implicit if one index is high and the other low. The indices $r, s = 1, \ldots, \mathcal{R}$, while Greek indices $\mu, \nu = 1, \ldots, m^2$ and also follow the summation convention.

2. Generalised algebras

Given an algebra $\mathcal{A}$ with a unit, a subspace of that algebra $\mathcal{B} \subset \mathcal{A}$ of finite dimension n is said to be a *generalised algebra* of rank $\mathcal{R}$ if for any basis $\{\lambda_a\}_{a=1,\ldots,n}$ of $\mathcal{B}$, there exist a $n^2 \times \mathcal{R}$ matrix of rank $\mathcal{R}$ given by (α_r^{ab}) such that

$$\alpha_r^{ab} \lambda_a \lambda_b \in \mathcal{B} \oplus \mathbb{I}, \tag{2.1}$$

where $\mathbb{I} = \operatorname{span}\{1\}$. Here a, b are summed over $1, \ldots, n$, $r = 1, \ldots, \mathcal{R}$, and $\mathcal{R} \leq n^2$. As stated in Section 1 throughout this article we shall assume that $\mathcal{A} = M_m(\mathbb{C})$. We can think of (2.1) simply as a set of relationships on the independent matrices $\{\lambda_a\}$. Alternatively we can think of a generalised algebra as an abstract vector space, with the only products defined being those defined by (2.1). The mapping that takes the elements of the generalised algebra into matrices can be thought of as a (matrix) representation of the underlying generalised algebra. In the same way as we think of Lie algebras and Clifford algebras as being the fundamental object, and the γ matrices as merely a representation.

The matrix algebra $M_m(\mathbb{C})$ comes equipped with an inner product

$$\langle f, g \rangle = \operatorname{tr}(f^\dagger g). \tag{2.2}$$

Since trace is defined we shall assume that all elements in $\mathcal{B}$ are traceless matrices. Since the inner product is positive definite its restriction onto $\mathcal{B}$ is also positive definite and so the matrix

$$g_{ab} = \langle \lambda_a, \lambda_b \rangle \tag{2.3}$$

is positive definite and Hermitian, $g_{ab} = \overline{g_{ba}}$. We label its inverse by g^{ab}, and define the elements $\{\lambda^a \in \mathcal{B}\}$ dual to $\{\lambda_b\}$ by

$$\lambda^a = g^{ba} \lambda_b, \quad \text{so } \langle \lambda^a, \lambda_b \rangle = \delta_b^a. \tag{2.4}$$

It is also useful to define the orthogonal projections onto, and perpendicular to $\mathcal{B}$

$$\eta : \mathcal{A} \mapsto \mathcal{B} \subset \mathcal{A}, \quad \eta(f) = \langle \lambda^a, f \rangle \lambda_a \tag{2.5}$$

$$\eta^\perp : \mathcal{A} \mapsto \mathcal{A}, \quad \eta^\perp(f) = f - \eta(f) - \frac{1}{m} \operatorname{tr}(f). \tag{2.6}$$

By taking the trace of (2.1) and its orthogonal projection onto $\mathcal{B}$, we get

$$\alpha_r^{ab} \left(\lambda_a \lambda_b - \langle \lambda^c, \lambda_a \lambda_b \rangle \lambda_c - \frac{1}{m} \operatorname{tr}(\lambda_a \lambda_b) \right) = 0, \tag{2.7}$$

$$\alpha_r^{ab} \eta^\perp(\lambda_a \lambda_b) = 0. \tag{2.8}$$

We shall see in Section 4 that it is useful to construct the $n^2 \times n^2$ projection matrix P^{ab}_{cd} of rank $\mathcal{R}$ so that we can write (2.1) as

$$P^{ab}_{cd}\left(\lambda_a\lambda_b - \langle\lambda^c, \lambda_a\lambda_b\rangle\lambda_c - \frac{1}{m}\mathrm{tr}(\lambda_a\lambda_b)\right) = 0. \tag{2.9}$$

For this we simply require an $n^2 \times \mathcal{R}$ matrix β^r_{ab} such that

$$\beta^r_{ab}\alpha^{ab}_s = \delta^r_s, \quad r, s = 1, \ldots, \mathcal{R}, \tag{2.10}$$

$$P^{ab}_{cd} = \sum_{r=1}^{\mathcal{R}} \alpha^{ab}_r \beta^r_{cd}. \tag{2.11}$$

The choice of $\mathcal{B}$ constrains, but does not completely determine β^r_{cd} and thus P^{ab}_{cd}. From (2.10) we see that there are $n^2(n^2 - \mathcal{R}) + \mathcal{R}^2$ linear constraints on the n^4 elements of P^{ab}_{cd}.

We note that for the given $\mathcal{B}$, $\mathcal{R}$ might not be maximal, i.e. there may exist other independent equations of the form (2.7) which we have chosen to ignore. Therefore we have the inequality

$$\dim(\mathrm{span}\{\{\lambda_a\lambda_b\}_{a,b=1,\ldots,n}, \{\lambda_a\}_{a=1,\ldots,n}, 1\}) \leq n^2 + n + 1 - \mathcal{R}. \tag{2.12}$$

3. The differential calculi: 1-forms

Let $\mathcal{A}$ be any unital associative $\star$-algebra. Of the many differential calculi which can be constructed over $\mathcal{A}$ the largest is the differential envelope or universal differential calculus $(\Omega^\star_u(\mathcal{A}), \mathrm{d}_u)$. Every other differential calculus can be considered as a quotient of it. For the definitions refer to, for example, [1,2,9, Chap. 6.1]. Let $(\Omega^\star(\mathcal{A}), \mathrm{d})$ be another differential calculus over $\mathcal{A}$. Then there exists a unique surjective d_u-homomorphism ϕ

$$\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{\mathrm{d}_u} & \Omega^1_u(\mathcal{A}) & \xrightarrow{\mathrm{d}_u} & \Omega^2_u(\mathcal{A}) & \xrightarrow{\mathrm{d}_u} & \cdots \\
\| & & \phi_1 \downarrow & & \phi_2 \downarrow & & \\
\mathcal{A} & \xrightarrow{\mathrm{d}} & \Omega^1(\mathcal{A}) & \xrightarrow{\mathrm{d}} & \Omega^2(\mathcal{A}) & \xrightarrow{\mathrm{d}} & \cdots
\end{array} \tag{3.1}$$

of $\Omega^\star_u(\mathcal{A})$ onto $\Omega^\star(\mathcal{A})$. It is given by

$$\phi(\mathrm{d}_u\xi) = \mathrm{d}\xi. \tag{3.2}$$

The restriction ϕ_p of ϕ to each Ω^p_u is defined by

$$\phi_p(f_0\,\mathrm{d}_u f_1 \cdots \mathrm{d}_u f_p) = f_0\,\mathrm{d} f_1 \cdots \mathrm{d} f_p. \tag{3.3}$$

making $\Omega^\star(\mathcal{A})$ a bimodule over $\mathcal{A}$.

Let us define $\Omega^1_{\mathcal{B}} = \Omega^1_{\mathcal{B}}(\mathcal{A})$, with respect to $\mathcal{B} \subset \mathcal{A}$ by requiring

$$\ker(\phi_1) = \left\{\sum_i f_i\,\mathrm{d}_u g_i \text{ with } f_i, g_i \in \mathcal{A} \,\middle|\, \sum_i f_i[h, g_i] = 0 \quad \forall h \in \mathcal{B}\right\}. \tag{3.4}$$

This is sufficient to define $\Omega^1_{\mathcal{B}}$. We define the set of derivations

$$\mathrm{Der}_{\mathcal{B}} = \{\mathrm{ad}(h) \mid h \in \mathcal{B}\}, \tag{3.5}$$

this is a complex vector space of dimension n. We now have the contraction given by

$$\begin{aligned} &\cdot : \Omega^1_{\mathcal{B}} \times \mathrm{Der}_{\mathcal{B}} \mapsto \mathcal{A}, \\ &\sum_i f_i \, \mathrm{d}g_i \cdot \mathrm{ad}(h) = \sum_i f_i [h, g_i], \end{aligned} \tag{3.6}$$

which satisfies

$$(f \xi g) \cdot X = f(\xi \cdot X) g \quad \forall f, g \in \mathcal{A}, \ \ \xi \in \Omega^1_{\mathcal{B}}, \ \ X \in \mathrm{Der}_{\mathcal{B}}. \tag{3.7}$$

From (3.4) we see that for $\xi \in \Omega^1_{\mathcal{B}}$ then $\xi \cdot X = 0$ for all $X \in \mathrm{Der}_{\mathcal{B}}$ implies $\xi = 0$. Thus there is an injective linear map from $\Omega^1_{\mathcal{B}}$ into the dual over $\mathcal{A}$ of $\mathrm{Der}_{\mathcal{B}}$:

$$\Omega^1_{\mathcal{B}} \hookrightarrow \mathrm{Der}^*_{\mathcal{B}} \stackrel{\mathrm{def}}{=} \{\xi : \mathrm{Der}_{\mathcal{B}} \mapsto \mathcal{A} \mid \xi \text{ is linear}\}. \tag{3.8}$$

We say that $\Omega^1_{\mathcal{B}}$ has *a co-frame* if $\Omega^1_{\mathcal{B}} = \mathrm{Der}^*_{\mathcal{B}}$.

Given the basis $\{\lambda_a\}_{a=1,\dots,n}$ of $\mathcal{B}$, we have the basis $\{e_a = \mathrm{ad}(\lambda_a)\}_{a=1,\dots,n}$ of $\mathrm{Der}_{\mathcal{B}}$. If $\Omega^1_{\mathcal{B}}$ has a co-frame then we can define the *co-frame* forms θ^b to be dual to e_a by

$$\theta^b \cdot e_a = \delta^a_b. \tag{3.9}$$

From (3.9) and (3.7) we have

$$\theta^a f = f \theta^a \quad \forall f \in \mathcal{A}. \tag{3.10}$$

We define the form θ to be

$$\theta \cdot \mathrm{ad}(h) = -h \quad \forall h \in \mathcal{B}, \tag{3.11}$$

which has the following identities:

$$-[\theta, f] = \mathrm{d}f \quad \forall f \in \mathcal{A}, \tag{3.12}$$

$$\theta = -\lambda_a \theta^a. \tag{3.13}$$

The relationship between these objects and those found in normal differential geometry are vague. The derivations $\{e_i\}$ are said to be analogous to the orthonormal frame for normal differential geometry whilst $\{\theta^a\}$ corresponds to its dual co-frame. There is no analogy to the form θ.

As already stated, in this article we shall take $\mathcal{A} = M_m(\mathbb{C})$ and $\mathcal{B} \subset \mathcal{A}$ as an n-dimensional subspace of traceless matrices. This is because of:

Theorem 1. *Assuming $\mathcal{A} = M_m(\mathbb{C})$ and $\mathcal{B} \subset \mathcal{A}$ is an n-dimensional subspace of traceless matrices then:*

– *There are exact expressions for θ^a and θ given by*

$$\theta^a = \gamma_\nu \lambda^{a\dagger}\, \mathrm{d}\gamma^{\nu\dagger}, \tag{3.14}$$

$$\theta = \frac{1}{m}\gamma_\mu\, \mathrm{d}\gamma^{\mu\dagger} = -\frac{1}{m}\, \mathrm{d}\gamma^{\mu\dagger}\gamma_\mu, \tag{3.15}$$

where $\{\gamma_\mu\}_{\mu=1,\dots,m^2}$ refer to any basis of $\mathcal{A} = M_m(\mathbb{C})$, and we set $\{\gamma^\nu\}_{\mu=1,\dots,m^2}$ to be its dual, so that $\langle\gamma^\nu, \gamma_\mu\rangle = \delta^\nu_\mu$.

– *$\Omega^1_{\mathcal{B}}$ has a co-frame.*

– *$\Omega^1_{\mathcal{B}}$ is a free module of rank n over $\mathcal{A}$, viz*

$$\Omega^1_{\mathcal{B}} = \otimes^n \mathcal{A}. \tag{3.16}$$

Before proving these we observe the following lemma.

Lemma 2.

$$\gamma_\mu f \gamma^{\mu\dagger} = \mathrm{tr}(f) \quad \forall f \in \mathcal{A}, \tag{3.17}$$

$$[f, \gamma_\mu g \otimes \gamma^{\mu\dagger}] = 0 \quad \forall f, g \in \mathcal{A}, \tag{3.18}$$

where $\{\gamma_\mu, \gamma^\nu\}_{\mu=1,\dots,m^2}$ as in Theorem 1.

Proof. First note that these are independent of the choice of basis γ_μ.

Now choose the basis $\{E_{ij}\}_{i,j=1,\dots,m}$ to be the matrix with a 1 in the ith row and the jth column, the natural basis for the $m \times m$ matrices, so that $E_{ij}E_{kl} = E_{il}\delta_{jk}$. These elements are orthonormal with respect to the trace inner product so E_{ij} is dual to itself. (During this proof indices i, j, k, l are summed from $1, \dots, m$.) Now let $f = f_{kl}E_{kl}$ with $f_{kl} \in \mathbb{C}$ so

$$E_{ij} f E_{ji} = E_{ij} f_{kl} E_{kl} E_{ji} = f_{kl}\delta_{jk}\delta_{jl}E_{ii} = f_{jj}E_{ii} = \mathrm{tr}(f). \tag{3.19}$$

Whilst

$$\begin{aligned} & f E_{ij} g \otimes E_{ji} - E_{ij} g \otimes E_{ji} f \\ &\quad = f_{kl}E_{kl}E_{ij} g \otimes E_{ji} - E_{ij} g \otimes E_{ji} f_{kl} E_{kl} \\ &\quad = f_{kl}(E_{kj}\delta_{li}\, g \otimes E_{ji} - E_{ij}\, g \otimes E_{jl}\delta_{ik}) \\ &\quad = f_{kl}(E_{kj} g \otimes E_{jl} - E_{kj} g \otimes E_{jl}) \\ &\quad = 0. \qquad \square \end{aligned} \tag{3.20}$$

Proof of Theorem 1. From (3.17) we have

$$\begin{aligned} \gamma_\nu \lambda^{a\dagger}\, \mathrm{d}\gamma^{\nu\dagger} &= \gamma_\nu \lambda^{a\dagger}(\gamma^{\nu\dagger}\theta - \theta\gamma^{\nu\dagger}) \\ &= (-\gamma_\nu\lambda^{a\dagger}\gamma^{\nu\dagger}\lambda_b + \gamma_\nu\lambda^{a\dagger}\lambda_b\gamma^{\nu\dagger})\theta^b \\ &= (-\mathrm{tr}(\lambda^{a\dagger})\lambda_b + \mathrm{tr}(\lambda^{a\dagger}\lambda_b)\theta^b \\ &= \langle\lambda^a, \lambda_b\rangle\theta^b = \delta^a_b\theta^b = \theta^a, \end{aligned} \tag{3.21}$$

hence (3.14). Also

$$\gamma_\mu d\gamma^{\mu\dagger} = \gamma_\mu[\lambda_a, \gamma^{\mu\dagger}]\theta^a = -m\lambda_a\theta^a = m\theta \tag{3.22}$$

whilst

$$0 = \mathrm{d}(\gamma_\mu \gamma^{\mu\dagger}) = \gamma_\mu \,\mathrm{d}\gamma^{\mu\dagger} + \mathrm{d}(\gamma_\mu)\gamma^{\mu\dagger}. \tag{3.23}$$

Thus (3.15). Given any linear map $\xi : \Omega^1_{\mathcal{B}} \mapsto \mathcal{A}$ then let $\xi_a = \xi \cdot e_a \in \mathcal{A}$ then $\xi = \xi_a \theta^a$ so $\Omega^1_{\mathcal{B}}$ has a co-frame, and is also a free module over $\mathcal{A}$ with rank n and basis $\{\theta^a\}$. □

The elements of $\Omega^1_u(\mathcal{A})$ which map onto θ^a and θ by the projection ϕ^1 are given by

$$\theta^a_u = \gamma_\mu \lambda^{a\dagger} \otimes \gamma^{\mu\dagger} = \gamma_\mu \lambda^{a\dagger} \,\mathrm{d}_u \gamma^{\mu\dagger}, \tag{3.24}$$

$$\theta_u = \frac{1}{m}\gamma_\mu \otimes \gamma^{\mu\dagger} - 1 \otimes 1 = \frac{1}{m}\gamma_\mu \,\mathrm{d}_u \gamma^{\mu\dagger}, \tag{3.25}$$

so

$$\phi^1(\theta^a_u) = \theta^a \quad \text{and} \quad \phi^1(\theta_u) = \theta. \tag{3.26}$$

As ϕ is not injective, θ^a_u and θ_u are not unique. However, $\theta^a_u = \eta^\star(\theta^a)$ (see example in Section 8.1), and θ_u does satisfy

$$-[\theta_u, f] = \mathrm{d}_u f, \tag{3.27}$$

which is shown by using (3.18).

Counterexamples. If one does not require that both $\mathcal{A} = M_m(\mathbb{C})$ and all the elements in $\mathcal{B}$ are traceless then the question of whether $\Omega^1_{\mathcal{B}}$ has a co-frame is nontrivial. Here are some examples where $\Omega^1_{\mathcal{B}}$ does not have a co-frame:
- $\mathcal{A}$ is Abelien.
- $\mathcal{A} = M_m(\mathbb{C})$ but $1 \in \mathcal{B}$. This is because $\mathrm{ad}(1) = 0$.
- $\mathcal{A} = \{$space of operators generated by x and p, where $[p, x] = 1\}$ and $\mathcal{B} = \mathrm{span}\{p, p^2, x\}$.

Whilst on the contrary $\Omega^1_{\mathcal{B}}$ does have a co-frame
- $\mathcal{A} = M_m(\mathbb{C})$ but $\mathcal{B} = \mathrm{span}\{1 + x, y, z\}$, where $\{x, y, z\}$ is a representation of $su(2)$.
- $\mathcal{A} = \{$space of operators generated by x and p where $[p, x] = 1\}$ and $\mathcal{B} = \mathrm{span}\{p, x\}$. This is the Heisenberg quantum algebra.

4. $\Omega^2_{\mathcal{B}}$ and generalised algebras

Having constructed the set of 1-forms $\Omega^1_{\mathcal{B}}$ we turn our attention to $\Omega^2_{\mathcal{B}}$, the structure of which is given by the following theorem.

Theorem 3. *Given* $\mathrm{d} : \Omega^1_{\mathcal{B}} \mapsto \Omega^2_{\mathcal{B}}$ *obeys* (3.1) *we have the following:*

$$\mathrm{d}\theta + \theta^2 = -\frac{1}{m}\mathrm{tr}(\lambda_a \lambda_b)\theta^a \theta^b, \tag{4.1}$$

$$\mathrm{d}\theta^a = -[\theta, \theta^a] - \langle \lambda^a, \lambda_b \lambda_c \rangle \theta^b \theta^c, \tag{4.2}$$

$$\eta^\perp(\lambda_a \lambda_b)\theta^a \theta^b = 0, \tag{4.3}$$

where $[\cdot,\cdot]$ is the graded commutator. If the contraction of 2-forms on pairs of vectors obeys the 2-form version of (3.7)

$$(f\xi g)\cdot(X,Y) = f(\xi\cdot(X,Y))g \qquad \forall f,g\in\mathcal{A},\ \xi\in\Omega^2_{\mathcal{B}},\ X,Y\in \mathrm{Der}_{\mathcal{B}}, \tag{4.4}$$

then either $\dim(\Omega^2_{\mathcal{B}})=0$ *or $\mathcal{B}$ is a generalised algebra. Let*

$$\theta^a\theta^b\cdot(e_c,e_d) = P^{ab}_{cd}. \tag{4.5}$$

Viewing P as an $n^2\times n^2$ matrix, if P has rank $\mathcal{R}$ then $\Omega^2_{\mathcal{B}}$ is a free module over $\mathcal{A}$ of rank $\mathcal{R}$., i.e.

$$\Omega^2_{\mathcal{B}} = \otimes^{\mathcal{R}}\mathcal{A}. \tag{4.6}$$

Proof. From (3.1) we have the standard relations on d given by

$$\mathrm{d}(\mathrm{d}f)=0 \quad\text{and}\quad \mathrm{d}(f\ \mathrm{d}h)=\mathrm{d}f\ \mathrm{d}h \qquad \forall f,h\in\mathcal{A}. \tag{4.7}$$

Using (3.15) we have

$$\begin{aligned}\mathrm{d}\theta &= \frac{1}{m}\,\mathrm{d}\gamma_\mu\,\mathrm{d}\gamma^{\mu\dagger} = \frac{1}{m}[\lambda_a,\gamma_\mu][\lambda_b,\gamma^{\mu\dagger}]\theta^a\theta^b\\ &= \frac{1}{m}(\gamma_\mu\lambda_a\gamma^{\mu\dagger}\lambda_b - \lambda_a\gamma_\mu\gamma^{\mu\dagger}\lambda_b - \gamma_\mu\lambda_a\lambda_b\gamma^{\mu\dagger} + \lambda_a\gamma_\mu\lambda_b\gamma^{\mu\dagger})\\ &= \left(-\lambda_a\lambda_b - \frac{1}{m}\mathrm{tr}(\lambda_a\lambda_b)\right)\theta^a\theta^b.\end{aligned} \tag{4.8}$$

Hence (4.1). From (3.14) we have

$$\begin{aligned}\mathrm{d}\theta^a &= \mathrm{d}(\gamma_\nu\lambda^{a\dagger}\,\mathrm{d}\gamma^{\nu\dagger}) = \mathrm{d}(\gamma_\nu\lambda^{a\dagger})\,\mathrm{d}\gamma^{\nu\dagger}\\ &= [\theta,\gamma_\nu\lambda^{a\dagger}][\theta,\gamma^{\nu\dagger}] = [\lambda_b,\gamma_\nu\lambda^{a\dagger}][\lambda_c,\gamma^{\nu\dagger}]\theta^b\theta^c\\ &= (\lambda_b\gamma_\nu\lambda^{a\dagger}\lambda_c\gamma^{\nu\dagger} - \gamma_\nu\lambda^{a\dagger}\lambda_b\lambda_c\gamma^{\nu\dagger} - \lambda_b\gamma_\nu\lambda^{a\dagger}\gamma^{\nu\dagger}\lambda_c + \gamma_\nu\lambda^{a\dagger}\lambda_b\gamma^{\nu\dagger}\lambda_c)\theta^b\theta^c\\ &= (\lambda_b\,\mathrm{tr}(\lambda^{a\dagger}\lambda_c) - \mathrm{tr}(\lambda^{a\dagger}\lambda_b\lambda_c) + \mathrm{tr}(\lambda^{a\dagger}\lambda_b)\lambda_c)\theta^b\theta^c\\ &= (\lambda_b\delta^a_c + \lambda_c\delta^a_b - \langle\lambda^a,\lambda_b\lambda_c\rangle)\theta^b\theta^c.\end{aligned} \tag{4.9}$$

Hence (4.2). From (4.1) we have

$$\begin{aligned}\mathrm{d}\theta &= -\,\mathrm{d}(\lambda_a\theta^a) = -\,\mathrm{d}\lambda_a\theta^a - \lambda_a\,\mathrm{d}\theta^a\\ &= [\theta,\lambda_a]\theta^a + \lambda_a(\theta\theta^a + \theta^a\theta + \langle\lambda^a,\lambda_b\lambda_c\rangle\theta^b\theta^c)\\ &= -2\theta^2 + \eta(\lambda_b\lambda_c)\theta^b\theta^c.\end{aligned} \tag{4.10}$$

Comparing this with (4.1) gives

$$-\theta^2 - \frac{1}{m}\mathrm{tr}(\lambda_a\lambda_b) = -2\theta^2 + \eta(\lambda_b\lambda_c)\theta^b\theta^c. \tag{4.11}$$

Hence (4.3). From (4.4) we have

$$\begin{aligned}P^{ab}_{cd}f &= \theta^a\theta^b\cdot(e_c,e_d)f = \theta^a\theta^b f\cdot(e_c,e_d)\\ &= f\theta^a\theta^b\cdot(e_c,e_d) = fP^{ab}_{cd} \quad \forall f\in\mathcal{A}.\end{aligned} \tag{4.12}$$

Hence P^{ab}_{cd} is in the centre of $\mathcal{A}$, so is a multiple of the unit element. Contracting this with (4.3) gives

$$\eta^{\perp}(P^{ab}_{cd}\lambda_a\lambda_b) = 0 \quad \text{with } P^{ab}_{cd} \in \mathbb{C}. \tag{4.13}$$

Thus either $\dim(\Omega^2_{\mathcal{B}}) = 0$ or $\mathcal{B}$ is a generalised algebra. Any element $\xi \in \Omega^2_{\mathcal{B}}$ can be written as

$$\begin{aligned} \xi &= \sum_\alpha f_0^{(\alpha)}\, \mathrm{d}f_1^{(\alpha)}\, \mathrm{d}f_2^{(\alpha)}, \quad \text{where } f_0^{(\alpha)}, f_1^{(\alpha)}, f_2^{(\alpha)} \in \mathcal{A} \\ &= \sum_\alpha f_0^{(\alpha)}[\lambda_a, f_1^{(\alpha)}][\lambda_b, f_2^{(\alpha)}]\theta^a\theta^b. \end{aligned} \tag{4.14}$$

Thus $\Omega^2_{\mathcal{B}}$ is a free module of $\mathcal{A}$, with a basis $\theta^a\theta^b$. From (4.5) we see that the number of independent sets of $\theta^a\theta^b$ is $\mathcal{R}$. □

In order to be consistent with Section 2 we shall assume that $P^2 = P$, thus

$$P^{ab}_{cd}\theta^c\theta^d = \theta^a\theta^b. \tag{4.15}$$

This is a special case of the results found in [3] where $F^a_{bc} = \frac{1}{2}P^{de}_{bc}\langle\lambda^a, \lambda_d\lambda_e\rangle$ and $K_{bc} = (1/2m)P^{de}_{bc}\,\mathrm{tr}(\lambda_d\lambda_e)$.

5. Higher-order forms

The higher forms are still free modules over $\mathcal{A}$ with the basis of $\Omega^p_{\mathcal{B}}$ being a quotient of the set

$$\{\theta^{a_1}\cdots\theta^{a_p}\}_{a_1,\ldots,a_p=1,\ldots,n}. \tag{5.1}$$

The quotient being given by the extension of (4.3) that adjacent pairwise contractions must vanish, viz

$$\eta^{\perp}(\lambda_{a_q}\lambda_{a_{q+1}})\theta^{a_1}\cdots\theta^{a_q}\theta^{a_{q+1}}\cdots\theta^{a_r} = 0 \quad \forall q = 1, \ldots, p-1. \tag{5.2}$$

For this to give a nontrivial free module it requires that there exist the $n^{p-2}\mathcal{R}(p-1)$ complex numbers

$$\{\varepsilon_{rt}^{a_1\cdots a_{p-2}} \in \mathbb{C}\}, \quad \text{where } a_1\cdots a_{p-2} = 1, \ldots, n,\ r = 1, \ldots, \mathcal{R},\ t = 1, \ldots, p-1, \tag{5.3}$$

such that

$$\sum_{r=1}^{\mathcal{R}} \alpha_r^{a_1a_2}\varepsilon_{r1}^{a_3\cdots a_p} = \sum_{r=1}^{\mathcal{R}} \alpha_r^{a_2a_3}\varepsilon_{r2}^{a_1a_4\cdots a_p} = \cdots = \sum_{r=1}^{\mathcal{R}} \alpha_r^{a_{p-1}a_p}\varepsilon_{r(p-1)}^{a_1\cdots a_{p-2}}$$
$$\forall a_1\cdots a_p = 1, \ldots, n,\quad r = 1, \ldots, \mathcal{R}. \tag{5.4}$$

The existence or otherwise of these ε's and hence the rank of $\Omega^p_{\mathcal{B}}$ depends on the nature of α_r^{ab}. However, if we do have a nontrivial $\Omega^p_{\mathcal{B}}$ then we have the extension of d given by the following theorem.

Theorem 4. *Given that* (5.2) *holds, the extension to* d *is given by*

$$\mathrm{d}: \Omega^\star_{\mathcal{B}} \mapsto \Omega^\star_{\mathcal{B}}, \qquad \mathrm{d}: \Omega^p_{\mathcal{B}} \mapsto \Omega^{p+1}_{\mathcal{B}}, \qquad \mathrm{d}\xi = -[\theta, \xi] + \chi(\xi), \tag{5.5}$$

where $[\cdot, \cdot]$ *is the graded commutator and*

$$\chi: \Omega^\star_{\mathcal{B}} \mapsto \Omega^\star_{\mathcal{B}}, \qquad \chi: \Omega^p_{\mathcal{B}} \mapsto \Omega^{p+1}_{\mathcal{B}}$$
$$\chi(f\theta^{a_1}\cdots\theta^{a_r}) = f\sum_{q=1}^{p}(-1)^{s+1}\langle\lambda^{a_q}, \lambda_b\lambda_c\rangle\theta^{a_1}\cdots\theta^{a_{q-1}}\theta^b\theta^c\theta^{a_{q+1}}\cdots\theta^{a_p}. \tag{5.6}$$

Both d *and* χ *are well defined and obey the graded Leibniz rule, i.e.*

$$\mathrm{d}(\zeta\xi) = \mathrm{d}(\zeta)\xi + (-1)^p\zeta\,\mathrm{d}(\xi) \quad \forall\zeta \in \Omega^p_{\mathcal{B}}, \ \ \xi \in \Omega^\star_{\mathcal{B}}, \tag{5.7}$$

d *obeys* (3.1) *and* χ *is left and right* $\mathcal{A}$ *linear.*

Proof. Now to show that they are well defined we note

$$\mathrm{d}(\eta^\perp(\lambda_a\lambda_b)\theta^a\theta^b) = 0 \tag{5.8}$$

since

$$\begin{aligned}
&\chi(\eta^\perp(\lambda_a\lambda_b)\theta^a\theta^b)\\
&\quad = \eta^\perp(\lambda_a\lambda_b)(\langle\lambda^a, \lambda_c\lambda_d\rangle\theta^c\theta^d\theta^b - \langle\lambda^b, \lambda_c\lambda_d\rangle\theta^a\theta^c\theta^d)\\
&\quad = \eta^\perp(\eta(\lambda_c\lambda_d)\lambda_b - \lambda_c\eta(\lambda_d\lambda_b))\theta^c\theta^d\theta^b\\
&\quad = \eta^\perp(\lambda_c\lambda_d\lambda_b - \eta^\perp(\lambda_c\lambda_d)\lambda_b\\
&\qquad - \mathrm{tr}(\lambda_c\lambda_d)\lambda_b - \lambda_c\lambda_d\lambda_b + \lambda_c\eta^\perp(\lambda_d\lambda_b) + \lambda_c\,\mathrm{tr}(\lambda_d\lambda_b))\theta^c\theta^d\theta^b\\
&\quad = 0,
\end{aligned} \tag{5.9}$$

whilst

$$[\theta, \eta^\perp(\lambda_a\lambda_b)\theta^a\theta^b] = 0, \tag{5.10}$$

thus d is well-defined on 2-forms. Higher forms follow from graded Leibniz. Also from graded Leibniz we have

$$\mathrm{d}(f_0\,\mathrm{d}f_1\cdots\mathrm{d}f_r) = \mathrm{d}f_0\,\mathrm{d}f_1\cdots\mathrm{d}f_r, \quad f_0, f_1, \ldots, f_r \in \mathcal{A}. \quad \square \tag{5.11}$$

6. An attempt at an alternative definition of $\Omega^2_{\mathcal{B}}$

It seems at first that the requirement that $\mathcal{B}$ be a generalised algebra is unnecessarily restrictive. Especially as this is not the case for most noncommutative versions of manifolds.

One idea is to examine the assumptions made and to see if weakening any of them would lead to a larger choice of $\mathcal{B}$. In this section we assume that $\Omega^2_{\mathcal{B}}$ is still bimodule of $\mathcal{A}$, but we do not require (4.4). Instead we impose the condition

$$f\theta^a\theta^b g \cdot (e_c, e_d) = fg\theta^a\theta^b \cdot (e_c, e_d) = fgQ^{ab}_{cd} \quad \forall f, g\mathcal{A}, \tag{6.1}$$

where $Q^{ab}_{cd} \in \mathcal{A}$. However, we find that if $\mathcal{B}$ is not a generalised algebra, then $\dim(\Omega^2_{\mathcal{B}}) = 0$ and we have gained nothing. To see this we first prove:

Lemma 5. *Given two sets of matrices $\{A^a, B^a \in M_m(\mathbb{C})\}_{a=1,\dots,n}$ such that*

$$\sum_{a=1}^{n} A^a C B^a = 0 \quad \forall C \in M_m(\mathbb{C}). \tag{6.2}$$

Then if the set $\{A^a\}$ are independent implies all $B^a = 0$.

Proof. Let the basis matrices be E_{ij} as in Lemma 2. With respect to this basis $A^a = A^a_{ij}E_{ij}$ (implicit sum on $ij\cdots$). Putting $C = E_{ij}$ in (6.2) gives

$$\begin{aligned} 0 &= \sum_a A^a E_{ij} B^a = \sum_a A^a_{kl} E_{kl} E_{ij} E_{pq} B^a_{pq} \\ &= \sum_a A^a_{kl} E_{kq} \delta_{li} \delta_{jp} B^a_{pq} = \sum_a A^a_{ki} B^a_{jq} E_{kq}, \end{aligned} \tag{6.3}$$

which since E_{kq} are independent gives

$$\sum_a A^a_{ki} B^a_{jq} = 0. \tag{6.4}$$

Multiplying by E_{ki} gives

$$0 = \sum_a A^a B^a_{jq}. \tag{6.5}$$

Thus implying $B^a = 0$ for all a. □

To prove the statement mentioned above, we proceed as follows. From (4.7) then for all $f \in \mathcal{A}$ we have

$$\begin{aligned} 0 &= \mathrm{d}\,\mathrm{d}f = \mathrm{d}([\lambda_a, f]\theta^a) \\ &= [\lambda_b, [\lambda_a, f]]\theta^b\theta^a - [\lambda_a, f](\theta\theta^a + \theta^a\theta + \mathrm{tr}(\lambda^a\lambda_b\lambda_c)\theta^b\theta^c) \\ &= \theta^2 f - \theta f\theta + \lambda_a f\theta\theta^a - f\lambda_a\theta\theta^a - f\theta^2 + \theta f\theta - \lambda_a f\theta\theta^a \\ &\quad + f\lambda_a\theta\theta^a - [\lambda_a\,\mathrm{tr}(\lambda^a\lambda_b\lambda_c), f]\theta^b\theta^c \\ &= [\theta^2 - \eta(\lambda_b\lambda_c)\theta^b\theta^c, f] \\ &= [\eta^\perp(\lambda_a\lambda_b), f]\theta^a\theta^b \\ &= \eta^\perp(\lambda_a\lambda_b)f\theta^a\theta^b \quad \forall f \in \mathcal{A}. \end{aligned} \tag{6.6}$$

Contracting with (e_c, e_d) gives

$$\eta^\perp(\lambda_a\lambda_b)fQ^{ab}_{cd} = 0 \tag{6.7}$$

from the contraction of (4.3) with (e_c, e_d). This is true for all $f \in \mathcal{A}$. Thus if $\mathcal{B}$ is not a generalised algebra, then all $\eta^{\perp}(\lambda_a\lambda_b)$ are independent, then from Lemma 5, $Q^{ab}_{cd} = 0$ and $\Omega^2_{\mathcal{A}}$ is trivial.

7. d-Homomorphisms of noncommutative algebras

Given any two subspaces $\mathcal{B}, \mathcal{B}' \subset \mathcal{A}$ and a linear map

$$\varphi : \mathcal{B} \mapsto \mathcal{B}'. \tag{7.1}$$

This generates the maps

$$\varphi_\star : \mathrm{Der}_{\mathcal{B}} \mapsto \mathrm{Der}_{\mathcal{B}'}, \qquad \varphi_\star(\mathrm{ad}(h)) = \mathrm{ad}(\phi(h)) \quad \forall h \in \mathcal{B}, \tag{7.2}$$

$$\varphi^\star : \Omega^\star_{\mathcal{B}'} \mapsto \Omega^\star_{\mathcal{B}}, \qquad \varphi^\star(\xi') \cdot X = \xi \cdot (\varphi_\star X) \quad \forall \xi' \in \Omega^\star\mathcal{B}', X \in \mathrm{Der}_{\mathcal{B}'}, \tag{7.3}$$

in a similar way to that of the push forward and pull back of differentiable maps between manifolds. However, unlike in commutative geometry the pullback map is not in general a d-homomorphism, i.e. it does not in general commute with the exterior derivative $\varphi^\star\,\mathrm{d}' \neq \mathrm{d}\varphi^\star$ where $\mathrm{d} : \Omega^\star_{\mathcal{B}} \mapsto \Omega^\star_{\mathcal{B}}$ and $\mathrm{d}' : \Omega^\star_{\mathcal{B}'} \mapsto \Omega^\star_{\mathcal{B}'}$.

There are some cases where they do commute. One simple case is when $\mathcal{B}'$ is a subspace of $\mathcal{B}$, if $\iota : \mathcal{B}' \hookrightarrow \mathcal{B}$ then $\mathrm{d}\iota^\star = \iota^\star\,\mathrm{d}'$. The set of relations on products of the basis elements $\{\lambda'_a \in \mathcal{B}'\}_{a=1,\ldots,n'}$ making $\mathcal{B}'$ into a generalised algebra are, of course, a subset of the relations on $\{\lambda_a\}$. However, since in our definition of a generalised algebra we give the possibility of ignoring some of these relationships we cannot say that the projection matrix from (2.9) P'^{ab}_{cd} for $\Omega^2_{\mathcal{B}'}$ is simply the restriction of P^{ab}_{cd} to $\mathcal{B}' \otimes \mathcal{B}'$.

7.1. Equivalent representations

Given $u \in GL_m(\mathbb{C})$, let

$$U : \mathcal{A} \mapsto \mathcal{A}, \qquad U : \mathcal{B} \mapsto \mathcal{B}', \qquad U(h) = uhu^{-1}. \tag{7.4}$$

This map is a bijective d-homomorphism, i.e. it preserves the generalised algebraic structure:

$$U(f+g) = U(f) + U(g) \quad \text{and} \quad U(fg) = U(f)U(g). \tag{7.5}$$

Hence, the $\alpha^{ab}_r = \alpha'^{ab}_r$, and U may be viewed as a map for one representation to another of the same generalised algebra. We shall call $\mathcal{B}$ and $\mathcal{B}'$ *equivalent* representations. It would be nice to have some idea if given two representations of the same generalised algebra whether they are equivalent. This gives rise to the following maps:

$$U_\star : \mathrm{Der}_{\mathcal{B}} \mapsto \mathrm{Der}_{\mathcal{B}'}, \qquad U_\star(\mathrm{ad}h) = \mathrm{ad}(U(h)) \tag{7.6}$$

$$\begin{aligned} &U^\star : \Omega^\star_{\mathcal{B}'} \mapsto \Omega^\star_{\mathcal{B}}, \\ &(U^\star\xi')\cdot(X_1,\ldots,X_r) = U^{-1}(\xi \cdot (U_\star X_1, \ldots, U_\star X_p)) \\ &\quad \forall \xi' \in \Omega^p_{\mathcal{B}'}, X_1, \ldots, X_p \in \mathrm{Der}_{\mathcal{B}}. \end{aligned} \tag{7.7}$$

Note the slightly different definition of $U^\star$. This map has the following properties:

Lemma 6. *If we choose the basis of $\mathcal{B}'$ to be $\lambda'_a = U(\lambda_a)$ then $\lambda^b = u^{-1\dagger}\lambda^b u^\dagger$ and $U^\star$ preserves the co-frame $U^\star(\theta'^a) = \theta^a$ and $U^\star(\theta') = \theta$. Furthermore if we choose the $\varepsilon^{ab\dots}$ to be equal so that*

$$\theta'^{a_1}\cdots\theta'^{a_p}\cdot(e'_{b_1}\cdots e'_{b_p}) = \theta^{a_1}\cdots\theta^{a_p}\cdot(e_{b_1}\cdots e_{b_p}), \tag{7.8}$$

then $U^\star$ preserves the exterior algebra and commutes with the exterior derivative:

$$U^\star(\xi'\zeta') = U^\star(\xi')U^\star(\zeta'), \tag{7.9}$$

$$U^\star(\mathrm{d}'\xi') = \mathrm{d}U^\star(\xi') \quad \forall \xi, \zeta' \in \Omega^\star_{\mathcal{B}}. \tag{7.10}$$

Proof. The preservation of the co-frame is trivial. Eq. (7.9) follows from (7.8). Now

$$\begin{aligned} U^\star[\theta', \xi'] &= U^\star(\theta'\xi' - (-1)^r\xi'\theta') = U^\star(\theta')U^\star(\xi') - (-1)^r U^\star(\xi')U^\star(\theta') \\ &= \theta U^\star(\xi') - (-1)^r U^\star(\xi')\theta = [\theta, U^\star(\xi')] \end{aligned} \tag{7.11}$$

and since $\langle\lambda'^a, \lambda'_b\lambda'_c\rangle = \langle\lambda^a, \lambda_b\lambda_c\rangle$ then

$$\begin{aligned} U^\star(\chi(f\theta'^{a_1}\dots\theta'^{a_p})) &= \frac{1}{2}U^\star(f)\sum_{q=1}^{p} U^\star(\langle\lambda'^a, \lambda'_b\lambda'_c\rangle)U^\star \\ &\quad \times (\theta'^{a_1}\cdots\theta'^{a_{q-1}}\theta'^b\theta'^c\theta'^{a_{q+1}}\cdots\theta'^{a_p}) \\ &= \frac{1}{2}U^\star(f)\sum_{q=1}^{p}\langle\lambda^a, \lambda_b\lambda_c\rangle\theta^{a_1}\cdots\theta^{a_{q-1}}\theta^b\theta^c\theta^{a_{q+1}}\cdots\theta^{a_p} \\ &= \chi(U^\star(f\theta'^{a_1}\cdots\theta'^{a_p})). \end{aligned} \tag{7.12}$$

Thus from (5.5) we have (7.10). □

7.2. The "Lie derivative"

As an aside we define a "derivative" $\mathcal{L}^\star_f : \Omega^p_{\mathcal{B}} \mapsto \Omega^p_{\mathcal{B}}$ for $f \in \mathcal{A}$. It is not obvious what role this function has. (It may be analogous to the Lie derivative in normal commutative geometry.) All that can be said about it is that it comes for free, i.e. we do not have to have any additional structure for its definition.

Let $\mathcal{B}' = U(\mathcal{B})$. As well as $U^\star$ there is another map from $\Omega^\star_{\mathcal{B}}$ to $\Omega^\star_{\mathcal{B}'}$. This is given by $\phi'\phi^{-1}$ where $\phi' : \Omega^\star_u \mapsto \Omega^\star_{\mathcal{B}'}$ is given by (3.1), and ϕ^{-1} is an $\mathcal{A}$-linear right inverse of ϕ given by

$$\begin{aligned} &\phi_p^{-1} : \Omega^p_{\mathcal{B}} \mapsto \Omega^p_u, \\ &\phi_p^{-1}(f\theta^{a_1}\dots\theta^{a_p}) = P^{a_1\dots a_p}_{b_1\dots b_p} f\theta^{b_1}_u\dots\theta^{b_p}_u, \end{aligned} \tag{7.13}$$

where $f \in \mathcal{A}$, θ^a_u is given by (3.24) and

$$P^{a_1\cdots a_p}_{b_1\cdots b_p} = \theta^{a_1}\cdots\theta^{a_p}\cdot(e_{b_1}\cdots e_{b_p}) \tag{7.14}$$

so that

$$\phi'_p\phi_p^{-1}\theta^{a_1}\cdots\theta^{a_p} = P^{a_1\cdots a_p}_{b_1\cdots b_p}\langle\lambda^{b_1}, \lambda_{c_1}\rangle\cdots\langle\lambda^{b_p}, \lambda_{c_p}\rangle\theta'^{c_1}\cdots\theta'^{c_p}. \tag{7.15}$$

These two maps are not equal. Given an $f \in \mathcal{A}$, let $U_t(g) = e^{tf} g e^{-tf}$. We now take the derivative of the map

$$\Omega^\star_{\mathcal{B}} \xrightarrow{\phi^{-1}} \Omega^\star_u \xrightarrow{\phi'} \Omega^\star_{\mathcal{B}'} \xrightarrow{U_t^\star} \Omega^\star_{\mathcal{B}}. \tag{7.16}$$

This is given by the "Lie" derivative

$$\mathcal{L}^\star_f : \Omega^\star_{\mathcal{B}} \mapsto \Omega^\star_{\mathcal{B}}, \qquad \mathcal{L}^\star_f(\xi) = \lim_{t\to 0} \frac{1}{t}(U^\star_t \circ \phi' \circ \phi^{-1}(\xi) - \xi), \tag{7.17}$$

which from (7.15) is given by

$$\begin{aligned}\mathcal{L}^\star_f(g\theta^{a_1}\cdots\theta^{a_p}) &= -[f,g]\theta^{a_1}\cdots\theta^{a_p} \\ &\quad - P^{a_1\cdots a_p}_{b_1\cdots b_p}\sum_{q=1}^{p}\langle\lambda^{b_q},[f,\lambda_c]\rangle\theta^{b_1}\cdots\theta^{b_{q-1}}\theta^c\theta^{b_{q+1}}\cdots\theta^{b_q}.\end{aligned} \tag{7.18}$$

We note that

$$U^\star(\mathcal{L}^\star_f\xi') = \mathcal{L}^\star_{U^\star f}(U^\star\xi') \quad \forall f \in \mathcal{A}, \xi' \in \Omega^\star_{\mathcal{B}'} \tag{7.19}$$

but that in general $\mathrm{d}\mathcal{L}^\star_f \neq \mathcal{L}^\star_f\,\mathrm{d}$.

8. Examples

We give here some examples of generalised algebras which have matrix representations and their corresponding noncommutative geometry.

8.1. Example: The universal algebra

Let $\mathcal{B}$ be given by $\mathcal{A}_0 \subset \mathcal{A}$, the subspace of all traceless matrices so that $\mathcal{A}_0 \oplus \mathbb{I} = \mathcal{A}$ and $n = m^2 - 1$. Since $\mathcal{A}$ is a matrix algebra then all derivations of $\mathcal{A}$ are in $\mathrm{Der}_{\mathcal{A}_0}$. Set $P^{ab}_{cd} = I_{(m^2-1)^2}$ the $(m^2-1)^2 \times (m^2-1)^2$ unit matrix. In this case the map $\phi : \Omega^\star_u \mapsto \Omega^\star_{\mathcal{A}_0}$ given by (3.1) is an isomorphism. To see this we can choose $\{\gamma_\mu\}_{\mu=0,\ldots,m^2-1}$ a basis for $\mathcal{A}$ by setting $\gamma_0 = 1$ and $\gamma_a = \lambda_a$ for $a = 1, \ldots, m^2-1$ as traceless matrices. In this basis ϕ_1 is given by

$$\phi_1\left(\sum_{\mu,\nu=0}^{m^2-1}\xi_{\mu\nu}\gamma_\mu \otimes \gamma_\nu\right) = \sum_{a,b=1}^{m^2-1}\xi_{ab}\lambda_a\,\mathrm{d}\lambda_b + \sum_{b=1}^{m^2-1}\xi_{0b}\,\mathrm{d}\lambda_b. \tag{8.1}$$

The inverse of this map can be calculated since

$$-\sum_{a,b=1}^{m^2-1}\xi_{ab}\lambda_a\lambda_b - \sum_{b=1}^{m^2-1}\xi_{0b}\lambda_b = \sum_{a=1}^{m^2-1}\xi_{a0}\lambda_a + \xi_{00}. \tag{8.2}$$

This is extended for all ϕ_p. The space of p-forms is now a free bimodule over $\mathcal{A}$ of rank $(m^2-1)^p$, and all the co-frame basis elements $\theta^{a_1}\cdots\theta^{a_p}$ are independent. The 1-form θ is given by θ_u in (3.25).

We can now view any other noncommutative geometry given by the subspace $\mathcal{B} \subset \mathcal{A}_0$ as being a sub-noncommutative geometry as stated in Section 7. The maps $\iota : \mathcal{B} \mapsto \mathcal{A}_0$ and $\eta : \mathcal{A}_0 \mapsto \mathcal{B}$ induce the pullbacks $\iota^\star : \Omega^\star_{\mathcal{A}_0} \mapsto \Omega^\star_{\mathcal{B}}$ and $\eta^\star : \Omega^\star_{\mathcal{B}} \mapsto \Omega^\star_{\mathcal{A}_0}$. By identifying $\Omega^\star_u$ and $\Omega^\star_{\mathcal{A}_0}$ then $\iota^\star = \phi_{\mathcal{B}} : \Omega^\star_u \mapsto \Omega^\star_{\mathcal{B}}$, and so, of course, commutes with d. It is easy to show that $\eta^\star(\theta^a) = \theta^a_u$ given by (3.24).

We can also view [9, Chap. 3] $\mathcal{A}_0$ as the fundamental representation of the Lie algebra $sl(m)$. For this we must choose the elements β^r_{ab} so that (8.6) below holds.

8.2. Example: The Lie algebra

A standard example of a generalised algebra is the case of a Lie algebra. This case has been studied in detail [9], especially when $\mathcal{B}$ is a representation of $su(2)$, which has been shown to be a noncommutative approximation to the sphere and $su(4)$ which is an analogue of the Euclidianised compactified Minkowski space.

If $\mathcal{B}$ is a representation of a Lie group of dimension n then

$$[\lambda_a, \lambda_b] = C^c_{ab}\lambda_c \in \mathcal{B} \tag{8.3}$$

for $a, b = 1, \ldots, n$, where C^c_{cd} are the structure constants. This make a total of $\frac{1}{2}n(n-1)$ independent equations. There are also the Casimir operators

$$\sum_{a=1}^{n'} \lambda'_i \lambda'_i \in \mathbb{I} \tag{8.4}$$

for any orthogonal basis $\{\lambda'_i\}_{i=1,\ldots,n'}$ of either $\mathcal{B}$ or any sub-Lie algebra $\mathcal{B}' \subset \mathcal{B}$. It is usual to ignore all these equations, and take only those given by (8.3). Thus the rank of the generalised algebra $\mathcal{R} = \frac{1}{2}n(n-1)$. Hence it is easier to replace r by the pair (c, d) with $c < d$. Thus

$$\alpha^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c, \quad \beta^{cd}_{ef} = \tfrac{1}{2}(\delta^c_e\delta^d_f - \delta^c_f\delta^d_e) \quad \text{for } c < d, \tag{8.5}$$

so

$$\theta^a\theta^b + \theta^b\theta^a = 0. \tag{8.6}$$

We also note that $h^\dagger \in \mathcal{B}$ for all $h \in \mathcal{B}$. Thus we can choose λ_a to be Hermitian, or anti-Hermitian. We get the same results if the λ_a's mutually commuted. We would then set $C^a_{bc} = 0$ in (8.3).

8.3. Example: q-Deformed algebra

A q-deformed algebra $\mathcal{A}$ is generated by the elements $x, y \in \mathcal{A}$, where $xy = qyx$. We can find an $M_m(\mathbb{C})$ representation for a q-deformed algebra if and only if $q^m = 1$. In order that $q \to 0$ as $m \to \infty$ let $q = \mathrm{e}^{2\pi \mathrm{i}/m}$. A representation is then given by

$$x = \left(\begin{array}{c|c} 0 & I_{m-1} \\ \hline 1 & 0 \end{array}\right), \qquad y = \mathrm{diag}(1, q, q^2, \dots, q^{m-1}), \tag{8.7}$$

where $I_{m-1} \in M_{m-1}(\mathbb{C})$ is the identity matrix. We see that x and y are nondegenerate, traceless matrices. Since $x^\dagger x = y^\dagger y = 1$ let

$$\lambda_1 = x, \quad \lambda_2 = y, \quad \lambda^1 = \frac{1}{m}x, \quad \lambda^2 = \frac{1}{m}y. \tag{8.8}$$

Explicit forms of θ^1, θ^2 are given in [3]. We note that $\mathrm{tr}(x^a y^b) = 0$ if either a or b is not a multiple of m. Thus we have

$$\langle \lambda^a, \lambda_b \lambda_c \rangle = 0, \quad \text{and} \quad \mathrm{tr}(\lambda_a \lambda_b) = 0 \qquad \forall a, b, c = 1, 2. \tag{8.9}$$

Eqs. (4.2) and (4.1) become

$$\mathrm{d}\theta^a = -[\theta, \theta^a], \tag{8.10}$$

$$\mathrm{d}\theta = -\theta^2. \tag{8.11}$$

The space of 2-forms $\Omega^2_{\mathcal{B}}$ is given by (4.3):

$$\lambda_1\lambda_1\theta^1\theta^1 + \lambda_2\lambda_1(q\theta^1\theta^2 + \theta^2\theta^1) + \lambda_2\lambda_2\theta^2\theta^2 = 0, \tag{8.12}$$

which implies that the rank of $\Omega^1_{\mathcal{B}} = 2$ and the rank of $\Omega^1_{\mathcal{B}} = 1$ with

$$\theta^1\theta^1 = \theta^2\theta^2 = 0, \quad q\theta^1\theta^2 + \theta^2\theta^1 = 0. \tag{8.13}$$

8.4. Example: The "fuzzy ellipsoid"

In the previous three examples the generalised algebra and associated noncommutative geometry were already well established. Here we give a simple generalised algebra of rank 1 which has not been studied before. We have called it the "fuzzy ellipsoid" since it is based on the fuzzy sphere with two of the three elements of $\mathcal{B}$ unchanged.

Let $\{J_1, J_2, J_3\}$ be an $M_m(\mathbb{C})$ Hermitian representation of $su(2)$ such that $[J_i, J_j] = i\varepsilon_{ijk}J_k$. Let

$$\mathcal{B} = \mathrm{span}\{\lambda_1, \lambda_2, \lambda_3\}, \tag{8.14}$$

where $\lambda_1 = -\mathrm{i}\kappa J_1$, $\lambda_2 = -\mathrm{i}\kappa J_2$, and

$$\mathrm{i}\lambda_3 = \alpha^{11}\lambda_1\lambda_1 + \alpha^{12}\lambda_1\lambda_2 + \alpha^{21}\lambda_2\lambda_1 + \alpha^{22}\lambda_2\lambda_2 - \tfrac{1}{12}\kappa^2 m(m^2-1)(\alpha^{11} - \alpha^{22}). \tag{8.15}$$

In this space we have $\dim_{\mathcal{A}}(\Omega^1_{\mathcal{B}}) = 3$ and $\dim_{\mathcal{A}}(\Omega^2_{\mathcal{B}}) = 1$ consisting of the span of the element

$$\frac{\theta^1\theta^1}{\alpha^{11}} = \frac{\theta^1\theta^2}{\alpha^{12}} = \frac{\theta^2\theta^1}{\alpha^{21}} = \frac{\theta^2\theta^2}{\alpha^{22}} \tag{8.16}$$

with $\theta^a\theta^b = 0$ otherwise.

The elements $\operatorname{tr}(\lambda_a\lambda_b)$ and $\langle\lambda^a, \lambda_b\lambda_c\rangle$ can be calculated. However, these are simply long sixth-order multipolynomials in m and α^{ab} which does not give any information.

9. Discussion

It would be nice to know which of the results discussed in this paper can be generalised to infinite-dimensional algebras, or alternatively to infinite-dimensional representation of generalised algebras. This is necessary for example in the q-deformed algebras when there does not exist an $m \in \mathbb{Z}$ such that $q^m = 1$, and for any representation of the Heisenberg algebra $[p, x] = i$. From the counterexamples (Section 3) one cannot assume that $\Omega^1_{\mathcal{B}}$ has a co-frame. Also since the trace of an operator is not in general defined this will cause further problems as Lemma 2 cannot be applied. We know from [3,4] that there will still exist restrictions on $\mathcal{B}$ equivalent to demanding that it is a generalised algebra.

There has been much work recently on the concept of linear connections and curvature in noncommutative geometry [3–7; 9, Chap. 3.5; 10] This work has been limited in the main to established algebras, such as Lie and q-deformed and quantum algebra. It would be nice to extend this work for generalised algebras.

As stated in Section 7 it would be nice to have some theorems (in line with those for Lie algebras) that ascertain when a generalised algebra has a matrix representation, and if given two such representation when they are equivalent.

Acknowledgements

The author would like to thank John Madore and Jihad Mourad for useful discussions which motivated this work. The author would also like to thank the Royal Society of London for a European Junior Fellowship, and Richard Kerner and the Laboratoire de Gravitation et Cosmologie Relativistes, Paris VI for their hospitality.

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